

Open subgroups of free topological groups

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Abstract

The theory of covering spaces is often used to prove the Nielson-Schreier theorem, which states that every subgroup of a free group is free. We apply the more general theory of semicovering spaces to obtain analogous subgroup theorems for topological groups: Every open subgroup of a free Graev topological group is a free Graev topological group. An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.

1 Introduction

A well-known application of covering space theory is the Nielson-Schreier theorem [19], which states that every subgroup of a free group is free [5, 11]. The corresponding situation for topological groups is more complicated since it is not true that every closed subgroup of a free topological group is free topological [7, 8, 9, 13]. The purpose of this paper is to use the theory of semicovering spaces developed in [4] to prove the following theorem.

Theorem 1. *Every open subgroup of a free Graev topological group is a free Graev topological group.*

Free topological groups are important objects in the general theory of topological groups and have an extensive literature dating back to their introduction by A.A. Markov [15] in the 1940s. Markov [15] defined the free topological group $F_M(X)$ on a space X and Graev [9] later introduced the free topological group $F_G(X, *)$ on a space X with basepoint $* \in X$. While the existence of these groups (for any X) follows abstractly from adjoint existence theorems [18], the theory of free topological groups has traditionally required the condition that X be completely regular since the canonical injections $\sigma : X \rightarrow F_M(X)$ and $\sigma_* : X \rightarrow F_G(X, *)$ are embeddings if and only if X is completely regular. In this paper, we exploit universal properties and adjoint functors (as opposed to working with complicated characterizations of the topology) to avoid placing any restrictions on X . For more on the theory of free topological groups, we refer the reader to [1, 20, 21].

Topological versions of the Nielson-Schreier Theorem [6, 16] have been attempted for free topological groups on Hausdorff k_ω -spaces, i.e. spaces which are the inductive limit of a sequence of compact subspaces. In this case, a subgroup of a free Graev topological group which admits a continuous Schreier traversal is free (Graev) topological. Theorem 1 is an improvement in the sense that it holds for the free topological group on an arbitrary topological space. Another difference between our approach and that in [6] is that we use topologically enriched graphs and categories as opposed to graphs and categories internal to a category of spaces.

Covering theoretic proofs of the algebraic Nielson-Schreier Theorem typically require an understanding of covering spaces and fundamental group(oid)s of graphs. Our proof of Theorem 1 generalizes this approach by replacing covering theory with the theory of semicoverings [4], graphs with **Top**-graphs (i.e. topological graphs with discrete vertex spaces), and the fundamental groupoid (fundamental group) with the fundamental **Top**-groupoid [4] (topological fundamental group [3]). Our application of the classification of semicoverings relies heavily on the fact that the theory applies to certain non-locally path connected spaces, called locally wep-connected spaces, which are not included in classical covering space theory.

This paper is structured as follows. In Section 2, we recall the basic theory of free topological groups and include a general comparison of the two notions of free topological groups (those in the sense of Graev and those in the sense of Markov). Using Theorem 1, we obtain a structure theorem for open subgroups of free Markov topological groups (Theorem 6): *An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.*

In Section 3, we extend the usual notion of an algebraic graph by allowing edge spaces to have non-discrete topologies; the resulting objects are called **Top**-graphs. We then present some universal constructions of topologically enriched categories and groupoids to be used in the computations of Section 4. In Section 4, we show the fundamental **Top**-groupoid (resp. topological fundamental group) of a **Top**-graph is a free **Top**-groupoid (resp. free Graev topological group). Finally, in Section 5, semicovering theory is applied to **Top**-graphs. Analogous to the fact that a covering of a graph is a graph, we find that a semicovering of a **Top**-graph is a **Top**-graph. The paper concludes with a proof of Theorem 1.

2 Free topological groups

Definition 2. Let X be a topological space. The *free Markov topological group on X* is the unique (up to isomorphism) topological group $F_M(X)$ equipped with a map $\sigma : X \rightarrow F_M(X)$ universal in the sense that every map $f : X \rightarrow G$ to a topological group G induces a unique, continuous homomorphism $\hat{f} : F_M(X) \rightarrow G$ such that $\hat{f}\sigma = f$.

The existence of free Markov topological groups is guaranteed by the Gen-

eral Adjoint Theorem [18]. In particular, if **Top** is the category of topological spaces and **TopGrp** is the category of topological groups, then $F_M : \mathbf{Top} \rightarrow \mathbf{TopGrp}$ is left adjoint to the forgetful functor $\mathbf{TopGrp} \rightarrow \mathbf{Top}$. Algebraically, $F_M(X)$ is the free group on the underlying set of X and $\sigma : X \rightarrow F_M(X)$ is the canonical injection of generators.

Definition 3. Let X be a space with basepoint $* \in X$. The *free Graev topological group* on $(X, *)$ is the unique (up to isomorphism) topological group $F_G(X, *)$ equipped with a map $\sigma_* : X \rightarrow F_G(X, *)$ such that $\sigma_*(*)$ is the identity element of $F_G(X, *)$ and universal in the sense that every map $f : X \rightarrow G$ to a topological group G which takes $*$ to the identity element of G induces a unique, continuous homomorphism $\tilde{f} : F_G(X, *) \rightarrow G$ such that $\tilde{f}\sigma_* = f$.

Similar to the unbased case, if \mathbf{Top}_* is the category of based topological spaces, then $F_G : \mathbf{Top}_* \rightarrow \mathbf{TopGrp}$ is left adjoint to the forgetful functor $\mathbf{TopGrp} \rightarrow \mathbf{Top}_*$. Here, the basepoint of a topological group is the identity element. Algebraically, $F_G(X, *)$ is the free group on the set $X \setminus \{*\}$, however, it is not necessarily isomorphic to $F_M(X \setminus \{*\})$ as a topological group. On the other hand, $F_M(X)$ is isomorphic to the free Graev topological group $F_G(X_+, *)$ where $X_+ = X \sqcup \{*\}$ has an isolated basepoint and $F_G(X, *)$ is isomorphic to the quotient topological group $F_M(X)/N$ where N is the conjugate closure of $\{*\}$.

Graev showed in [9, Theorem 2] that the isomorphism class of $F_G(X, *)$ as a topological group does not depend on the choice of basepoint, i.e. given any other point $*' \in X$ there is an isomorphism $F_G(X, *) \rightarrow F_G(X, *)'$ of topological groups.¹ It remains to understand when $F_G(X, *)$ is isomorphic to the free Markov topological group $F_M(Y)$ on some space Y . This is answered by Graev in [9] in the case that X is completely regular; we modify Graev's argument only slightly in order to obtain a more general result.

Lemma 4. *If X is the disjoint union $X = A_1 \sqcup A_2$ of open sets $A_i \subset X$ and $e_i \in A_i$, then $F_G(X, e_1)$ is isomorphic to the free Markov topological group $F_M(A_1 \vee A_2)$ on the wedge sum $A_1 \vee A_2 = X/\{e_1, e_2\}$.*

Proof. Let $q : X \rightarrow A_1 \vee A_2$ be the quotient map making the identification $q(e_1) = z = q(e_2)$. Define a map $f : X \rightarrow F_M(X)$ by $f(a) = ae_2^{-1}$ for $a \in A_1$ (here $a_1e_2^{-1}$ is the product in $F_M(X)$) and $f(a) = a$ for $a \in A_2$. Since $F_M(q)(e_1) = F_M(q)(e_2)$, the composition $\psi = F_M(q)f : X \rightarrow F_M(A_1 \vee A_2)$ takes e_1 to the identity of $F_M(A_1 \vee A_2)$ and induces a continuous homomorphism $\tilde{\psi} : F_G(X, e_1) \rightarrow F_M(A_1 \vee A_2)$. Note that $\tilde{\psi}(e_2) = z$.

Now consider the map $g : X \rightarrow F_G(X, e_1)$ where $g(a) = ae_2$, $a \in A_1$ is the product in $F_G(X, e_1)$ and $g(a) = a$, $a \in A_2$. Since e_1 is the identity in $F_G(X, e_1)$, $g(e_1) = e_1e_2 = e_2 = g(e_2)$. We obtain a continuous map $\phi : A_1 \vee A_2 \rightarrow F_G(X, e_1)$ on the quotient such that $\phi(z) = e_2$ and which induces a continuous homomorphism $\hat{\phi} : F_M(A_1 \vee A_2) \rightarrow F_G(X, e_1)$.

¹Graev assumes X is completely regular, however, the argument given also applies to the general case.

A direct check shows that $\hat{\phi}\tilde{\psi}$ is the identity homomorphism of $F_G(X, e_1)$ and $\tilde{\psi}\hat{\phi}$ is the identity of $F_M(A_1 \vee A_2)$. In particular, if $a \in A_1 \setminus \{e_1\}$, then $\hat{\phi}\tilde{\psi}(a) = \hat{\phi}(ae_2^{-1}) = \hat{\phi}(a)\hat{\phi}(e_2)^{-1} = (ae_2)e_2^{-1} = a$ and $\tilde{\psi}\hat{\phi}(a) = \tilde{\psi}(ae_2) = \tilde{\psi}(a)\tilde{\psi}(e_2) = ae_2^{-1}e_2 = a$. The other cases are straightforward and left to the reader. \square

Theorem 5. *For any space X , the following are equivalent.*

1. X is connected.
2. $F_G(X, *)$ is connected.
3. $F_G(X, *)$ is not isomorphic to a free Markov topological group.

Proof. 1. \Rightarrow 2. Suppose X is connected and let C be the connected component of the identity in $F_G(X, *)$. Since $\sigma_* : X \rightarrow F_G(X, *)$ is continuous, the generating set $\sigma_*(X)$ is a connected subspace of $F_G(X, *)$ containing $*$ and is therefore contained in C . The connected component of the identity element in a general topological group is a subgroup [1, 1.4.26]. Therefore $C = F_G(X, *)$.
 2. \Rightarrow 3. Every free Markov topological group is disconnected since the canonical map $X \rightarrow *$ collapsing X to a point induces a continuous homomorphism $F_M(X) \rightarrow F_M(*) = \mathbb{Z}$ onto to the discrete group of integers. Therefore, if $F_G(X, *)$ is connected, $F_G(X, *)$ cannot be isomorphic to a free Markov topological group.
 3. \Rightarrow 1. This follows directly from Lemma 4. \square

Combining Theorems 1 and 5 and the fact that every free Markov topological group is a free Graev topological group, we obtain a structure theorem for open subgroups of free Markov topological groups. This result generalizes that in [6] for free Markov topological groups on Hausdorff k_ω -spaces.

Theorem 6. *An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.*

3 Topologically enriched graphs and categories

The rest of this paper is devoted to a proof of Theorem 1.

3.1 Top-graphs

A **Top-graph** Γ consists of a discrete space of vertices Γ_0 , an edge space Γ , and continuous structure maps $\partial_0, \partial_1 : \Gamma \rightarrow \Gamma_0$. For convenience, we sometimes let Γ denote the **Top-graph** itself. The set of composable edges in Γ is the pullback $\Gamma \times_{\Gamma_0} \Gamma = \{(e, e') | \partial_1(e) = \partial_0(e')\}$.

For each pair of vertices $x, y \in \Gamma_0$, let $\Gamma_x = \partial_0^{-1}(x)$, $\Gamma^y = \partial_1^{-1}(y)$, and $\Gamma(x, y) = \Gamma_x \cap \Gamma^y$. Since we require the vertex space of a **Top-graph** to be discrete, the edge space decomposes as the topological sum $\Gamma = \coprod_{(x,y) \in X \times X} \Gamma(x, y)$ over ordered pairs of vertices.

Since it is possible that both $\Gamma(x, y)$ and $\Gamma(y, x)$ are non-empty, we are motivated to make the following construction. Let $\Gamma(x, y)^{-1}$ denote a homeomorphic copy of $\Gamma(x, y)$ for each pair $(x, y) \in \Gamma_0 \times \Gamma_0$. Here $e \in \Gamma(x, y)$ corresponds to $e^{-1} \in \Gamma(x, y)^{-1}$. Define a new **Top**-graph Γ^\pm to have vertex spaces Γ_0 and $\Gamma^\pm(x, y) = \Gamma(x, y) \sqcup \Gamma(y, x)^{-1}$. In particular, note that $\Gamma^\pm(x, x) = \Gamma(x, x) \sqcup \Gamma(x, x)^{-1}$.

A morphism $f : \Gamma \rightarrow \Gamma'$ of **Top**-graphs consists of a pair of continuous functions $(f_0, f) : (\Gamma_0, \Gamma) \rightarrow (\Gamma'_0, \Gamma')$ such that $\partial'_i \circ f = f_0 \circ \partial_i$, $i = 1, 2$. Such a morphism is said to be *quotient* if f_0 and f are quotient maps of spaces (note f_0 only needs to be surjective to be quotient). There is also an obvious notion of sub-**Top**-graph $S \subseteq \Gamma$. We say such a sub-**Top**-graph is *wide* if $S_0 = \Gamma_0$. The category of **Top**-graphs is denoted **TopGraph**.

Definition 7. The *geometric realization* of a **Top**-graph Γ is the topological space

$$|\Gamma| = \Gamma_0 \sqcup (\Gamma \times [0, 1]) / \sim \text{ where } \partial_i(\alpha) \sim (\alpha, i) \text{ for } i = 0, 1.$$

A **Top**-graph Γ is *connected* if $|\Gamma|$ is path connected, or equivalently, if for each $x, y \in \Gamma_0$, there is a sequence of vertices $x = a_1, a_2, \dots, a_n = y$ such that $\Gamma^\pm(a_j, a_{j+1}) \neq \emptyset$ for $j = 1, \dots, n-1$.

We typically assume **Top**-graphs are connected.

Remark 8. For any $0 < r < 1$, the image of $\Gamma_x \times [0, r]$ in the quotient $|\Gamma|$ is homeomorphic to the cone CX on X . Similarly, if $Z = \Gamma(x, y) \sqcup \Gamma(y, x)$, then the image of $Z \times [0, 1]$ in the $|\Gamma|$ is the unreduced suspension SZ . Finally, note

$$|\Gamma| \setminus \Gamma_0 = \coprod_{(x,y)} (\Gamma(x, y) \times (0, 1)).$$

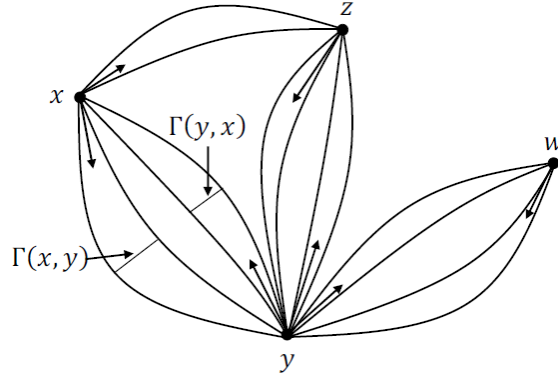


Figure 1: The realization of a **Top**-graph Γ with four vertices. Here $\Gamma(z, x)$, $\Gamma(w, z)$, and $\Gamma(z, w)$ are all empty.

Remark 9. It is unfortunate that $|\Gamma|$ need not be first countable at its vertices, however, it is possible to change the topology on $|\Gamma|$ without changing homotopy type so that each vertex has a countable neighborhood base. The *vertex neighborhood* of $x \in \Gamma_0$ of radius $r \in (0, 1)$ is the image of

$$(\Gamma_x \times [0, r)) \cup (\Gamma^x \times (1 - r, 1])$$

and is denoted $B(x, r)$. An *edge neighborhood* of a point $(e, t) \in \Gamma(x, y) \times (0, 1)$ is the homeomorphic image of a set $U \times (a, b)$ where U is an open neighborhood of e in $\Gamma(x, y)$ and $0 < a < t < b < 1$. The basis consisting of vertex and edge neighborhoods is closed under finite intersection and generates a topology which may be strictly coarser than the quotient topology (but only at vertices). Note that each vertex neighborhood $B(x, r)$ is contractible onto x and the set $\{B(x, 1/n) | n \geq 1\}$ is a countable neighborhood base at x .

From now on, we assume $|\Gamma|$ has the courser topology generated by vertex and edge neighborhoods.

Example 10. If Γ is a **Top**-graph with a single vertex, then $|\Gamma|$ is the *generalized wedge of circles* $\Sigma(\Gamma_+)$ (where Σ denotes reduced suspension) studied in detail in [2]. When $\Gamma_0 = \{x_0, x_1\}$ and the structure maps are the two constant maps $\partial_i : \Gamma \rightarrow \Gamma_0$, $\partial_i(\alpha) = x_i$ (equivalently, $\Gamma = \Gamma(x_0, x_1)$), then $|\Gamma|$ is the unreduced suspension $S\Gamma$. Thus, unlike a discrete graph, a **Top**-graph may be simply connected but not contractible, e.g. $\Gamma = S^1$.

Path component spaces 11. We end this section with a useful construction on **Top**-graphs: The *path component space* of a topological space X is the quotient space $\pi_0(X)$ where each path component is identified to a point. If Γ is a **Top**-graph, then $\pi_0(\Gamma)$ is the **Top**-graph with vertex space Γ_0 and $\pi_0(\Gamma)(x, y) = \pi_0(\Gamma(x, y))$. The canonical quotient morphism of **Top**-graphs $q : \Gamma \rightarrow \pi_0(\Gamma)$ is the identity on vertices and takes an edge e to its path component $[e]$.

Remark 12. Another useful construction is a section to the path component functor $\pi_0 : \mathbf{TopGraph} \rightarrow \mathbf{TopGraph}$. Given any space X , there a (paracompact Hausdorff) space $h(X)$ and a natural homeomorphism $\pi_0(h(X)) \cong X$ [10]. Thus for any **Top**-graph Γ , we define $h(\Gamma)$ to have object space Γ_0 and $h(\Gamma)(x, y) = h(\Gamma(x, y))$ so that $\pi_0(h(\Gamma)) \cong \Gamma$.

3.2 Top-categories and qTop-categories

Our use of enriched categories aligns with that in [14]. If a **Top**-graph C comes equipped with continuous composition map $C \times_{C_0} C \rightarrow C$ making C a category in the usual way, then C is a **Top-category** (or a category enriched over **Top**). Since $Ob(C) = C_0$ is discrete, $C \times_{C_0} C$ decomposes as a topological sum of products $C(x, y) \times C(y, z)$. Thus to specify a **Top-category** one only need specify the hom-spaces $C(x, y)$ and continuous composition maps $C(x, y) \times C(y, z) \rightarrow C(x, z)$. If composition maps are only continuous in each variable, then C is an **sTop-category** (the "s" is for "semitopological" as in [1]). A **Top-functor**

$F : \mathcal{C} \rightarrow \mathcal{D}$ of **Top**-categories is a functor such that each function $F_{x,y} : C(x, y) \rightarrow D(F(x), F(y))$ is continuous. The category of **Top**-categories and **Top**-functors is denoted **TopCat**.

An *involution* on a small category C is a function $C \rightarrow C$ defined by functions $C(x, y) \rightarrow C(y, x)$, $f \mapsto f^*$ such that $(f^*)^* = f$, $(fg)^* = g^*f^*$, and $(id_x)^* = id_x$. A **Top**-category (resp. a **sTop**-category) equipped with a continuous involution is a **Top**-category with continuous involution (resp. a **qTop**-category). If \mathcal{G} is a **Top**-category (resp. **sTop**-category) whose underlying category is a groupoid and the involution given by the inversion functions $\mathcal{G}(x, y) \rightarrow \mathcal{G}(y, x)$ is continuous, then \mathcal{G} is a **Top**-groupoid (resp. **qTop**-groupoid).

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of categories with involution *preserves involution* if $F(f^*) = F(f)^*$. In particular, a **qTop**-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of **qTop**-categories is an involution preserving functor which is continuous on hom-spaces.

The notion of **qTop**-groupoid is particularly relevant and is studied in Section 4 of [4]. The following Lemma is a useful fact asserting that the category **TopGrpd** of **Top**-groupoids is a full reflective subcategory of the category **qTopGrpd** of **qTop**-groupoids.

Lemma 13. [4, Lemma 4.5] *The forgetful functor $\mathbf{TopGrpd} \rightarrow \mathbf{qTopGrpd}$ has a left adjoint $\tau : \mathbf{qTopGrpd} \rightarrow \mathbf{TopGrpd}$ which is the identity on the underlying groupoids and functors.*

Free Top-categories 14. The *free Top*-category generated by a **Top**-graph Γ is the **Top**-category $\mathcal{C}(\Gamma)$ with object space Γ_0 and in which morphisms are finite sequences $e_1 e_2 \dots e_n$ of composable edges $e_i \in \Gamma$. In particular, the hom-space $\mathcal{C}(\Gamma)(x, y)$ is topologized as the topological sum $\coprod \Gamma(x, a_1) \times \Gamma(a_1, a_2) \times \dots \times \Gamma(a_n, y)$ where the sum ranges over all finite sequences a_1, \dots, a_n in Γ_0 . In order to obtain a category, we add an isolated identity morphism $\{id_x\}$ to each space $\mathcal{C}(\Gamma)(x, x)$. Note this construction yields a functor $\mathcal{C} : \mathbf{TopGraph} \rightarrow \mathbf{TopCat}$ left adjoint to the forgetful functor $\mathbf{TopCat} \rightarrow \mathbf{TopGraph}$.

The construction of $\mathcal{C}(\Gamma)$ is easily modified to include a continuous involution. In particular, the free **Top**-category with (continuous) involution on Γ is $\mathcal{C}^\pm(\Gamma) = \mathcal{C}(\Gamma^\pm)$, the free-**Top** category on the **Top**-graph Γ^\pm described in the previous section. Thus a generic (non-identity) morphism of $\mathcal{C}^\pm(\Gamma)$ may be given by a sequence $e_1^{\delta_1} e_2^{\delta_2} \dots e_n^{\delta_n}$ where $e_i \in \Gamma$, $\delta_i \in \{\pm 1\}$.

Remark 15. The construction of the path component **Top**-graph $\pi_0(\Gamma)$ also applies to **Top**-categories. Recall that for spaces X, Y , there is a canonical, continuous bijection $\psi : \pi_0(X \times Y) \rightarrow \pi_0(X) \times \pi_0(Y)$ which is not necessarily a homeomorphism [2]. Consequently, if Γ is a **Top**-category (with continuous involution), then $\pi_0(\Gamma)$ naturally inherits the structure of a **sTop**-category (**qTop**-category) but is not always a **Top**-category (with continuous involution). While it is possible to avoid this difficulty by restricting to a cartesian closed category of spaces, we remain in the usual topological category in order to prove Theorem 1 in full generality.

The observation on products in the previous remark immediately extends to the following lemma.

Lemma 16. *Given a **Top**-graph Γ , there is a canonical **qTop**-functor $\psi : \pi_0(\mathcal{C}^\pm(\Gamma)) \rightarrow \mathcal{C}^\pm(\pi_0(\Gamma))$ given by $[e_1^{\delta_1} e_2^{\delta_2} \dots e_n^{\delta_n}] \mapsto [e_1]^{\delta_1} [e_2]^{\delta_2} \dots [e_n]^{\delta_n}$ which is an isomorphism of the underlying categories.*

Free Top-groupoids 17. Given a **Top**-graph Γ , the free **Top**-groupoid generated by Γ is denoted $\mathcal{F}(\Gamma)$ and is characterized by the following universal property: If \mathcal{G} is a **Top**-groupoid any **Top**-graph morphism $f : \Gamma \rightarrow \mathcal{G}$ extends uniquely to a **Top**-functor $\hat{f} : \mathcal{F}(\Gamma) \rightarrow \mathcal{G}$. In other words, $\mathcal{F} : \mathbf{TopGraph} \rightarrow \mathbf{TopGrpd}$ is left adjoint to the forgetful functor $\mathbf{TopGrpd} \rightarrow \mathbf{TopGraph}$.

The underlying groupoid of $\mathcal{F}(\Gamma)$ is simply the free groupoid generated by the underlying algebraic graph of Γ , i.e. $Ob(\mathcal{F}(\Gamma)) = \Gamma_0$ and a morphism is a reduced word $e_1^{\delta_1} e_2^{\delta_2} \dots e_n^{\delta_n} \in \mathcal{C}^\pm(\Gamma)$. See [5, 12] for more on free groupoids. The topological structure of $\mathcal{F}(\Gamma)$ is characterized as follows: Let $\mathcal{F}_R(\Gamma)$ be the free groupoid on the underlying algebraic graph of Γ which is the quotient of $\mathcal{C}^\pm(\Gamma)$ with respect to the word reduction functor $R : \mathcal{C}^\pm(\Gamma) \rightarrow \mathcal{F}_R(\Gamma)$. Note that $\mathcal{F}_R(\Gamma)$ is a **qTop**-groupoid. The free **Top**-groupoid is the τ -reflection

$$\mathcal{F}(\Gamma) = \tau(\mathcal{F}_R(\Gamma)).$$

It is straightforward to verify that this groupoid has the desired universal property. In the case that Γ has a single vertex [2], $\mathcal{C}^\pm(\Gamma)$ is the free topological monoid with continuous involution on Γ and $\mathcal{F}(\Gamma)$ is the free topological group $F_M(\Gamma) = F_G(\Gamma_+, *)$.

3.3 Vertex groups and free topological groups

We now show each vertex group of a free **Top**-groupoid is a free Graev topological group.

Definition 18. A **Top**-graph T is a *tree* if T is discrete and $|T|$ is contractible. If Γ is a **Top**-graph, a tree $T \subseteq \Gamma$ is *maximal* in Γ if $T_0 = \Gamma_0$. Even though T is itself a discrete **Top**-graph, the edge space T need not be open in Γ . A *tree groupoid* is a groupoid \mathcal{G} such that each set $\mathcal{G}(x, y)$ has exactly one element. Note that if T is a tree, then $\mathcal{F}(T)$ is a discrete tree groupoid.

The standard argument that every graph contains a maximal tree is the same for **Top**-graphs.

Lemma 19. *Every connected **Top**-graph contains a maximal tree.*

Fix a **Top**-graph Γ , a maximal tree $T \subset \Gamma$, and a vertex $v \in \Gamma_0$. Let $\mathcal{F}(\Gamma)(v) = \mathcal{F}(\Gamma)(v, v)$ be the vertex topological group at v . Recall that if Γ has a single vertex v , then $\mathcal{F}(\Gamma) = \mathcal{F}(\Gamma)(v) \cong F_M(\Gamma) \cong F_G(\Gamma_+, *)$. Therefore, we restrict to the case when Γ has more than one vertex. In this case, T has non-empty edge space.

For vertex $x \in \Gamma_0$, let $\gamma_{v,x}$ be the unique element of $\mathcal{F}(T)(v, x)$. Define a retraction $r_T : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma)(v)$ of groupoids so that if $\alpha \in \mathcal{F}(\Gamma)(x, y)$, then $r_T(\alpha) = \gamma_{v,x} \alpha \gamma_{y,v}$. By definition, if $i : \mathcal{F}(T)(v) \rightarrow \mathcal{F}(\Gamma)$ is the inclusion of the

vertex group, then $r_T i$ is the identity of $\mathcal{F}(\Gamma)(v)$. In fact, since composition in $\mathcal{F}(\Gamma)$ is continuous, we have a **Top**-functor.

Lemma 20. $r_T : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma)(v)$ is a retraction of **Top**-groupoids.

Let $\sigma : \Gamma \rightarrow \mathcal{F}(\Gamma)$ be the canonical **Top**-graph morphism. It is known that the underlying group of $\mathcal{F}(\Gamma)(v)$ is freely generated by the set $r_T \sigma(\Gamma \setminus T)$ (See, for instance, [5, 8.2.3]).

Note that if $\gamma \in \Gamma$, then $r_T \sigma(\gamma)$ is the identity element of the group $\mathcal{F}(\Gamma)(v)$ if and only if $\gamma \in T$. Let Γ/T be the quotient of the edge space Γ and choose the basepoint $*$ in Γ/T to be the image of T . Thus the function $r_T \sigma : \Gamma \rightarrow \mathcal{F}(\Gamma)(v)$ induces a continuous injection $s : \Gamma/T \rightarrow \mathcal{F}(\Gamma)(v)$ such that $s q = r_T \sigma$ and where $s(*)$ is the identity element of $\mathcal{F}(\Gamma)$. Since $r_T \sigma(\Gamma \setminus T)$ freely generates $\mathcal{F}(\Gamma)(v)$, the continuous group homomorphism $\tilde{s} : F_G(\Gamma/T, *) \rightarrow \mathcal{F}(\Gamma)(v)$ induced by s is an isomorphism of groups.

Theorem 21. If Γ has more than one vertex and $T \subset \Gamma$ is a maximal tree, then the vertex group $\mathcal{F}(\Gamma)(v)$ is isomorphic to the free Graev topological group $F_G(\Gamma/T, *)$.

Proof. It suffices to show the inverse of $\tilde{s} : F_G(\Gamma/T, *) \rightarrow \mathcal{F}(\Gamma)(v)$ is continuous. Let $\sigma_* : \Gamma/T \rightarrow F_G(\Gamma/T, *)$ be the inclusion of generators and $q : \Gamma \rightarrow \Gamma/T$ be the quotient map. The composition $g = \sigma_* q : \Gamma \rightarrow F_G(\Gamma/T, *)$ may be viewed as a morphism of **Top**-graphs taking all vertices of Γ to the unique vertex of $F_G(\Gamma/T, *)$. Since $F_G(\Gamma/T, *)$ is a **Top**-groupoid, there is a unique **Top**-functor $\hat{g} : \mathcal{F}(\Gamma) \rightarrow F_G(\Gamma/T, *)$ such that $\hat{g} \sigma = g$. If $i : \mathcal{F}(\Gamma)(v) \rightarrow \mathcal{F}(\Gamma)$ is the inclusion of the vertex group, the composition $\hat{g} i : \mathcal{F}(\Gamma)(v) \rightarrow F_G(\Gamma/T, *)$ is continuous. We check that $\hat{g} i$ is the inverse of \tilde{s} .

Recall that $s q = r_T \sigma$. Therefore

$$\tilde{s} \hat{g} \sigma = \tilde{s} g = \tilde{s} \sigma_* q = s q = r_T \sigma.$$

Uniqueness of extensions then gives $\tilde{s} \hat{g} = r_T$.

$$\begin{array}{ccccc} & & \Gamma & & \\ & \swarrow \sigma & \downarrow g & \searrow q & \\ \mathcal{F}(\Gamma) & \xrightarrow{\hat{g}} & F_G(\Gamma/T, *) & \xleftarrow{\sigma_*} & \Gamma/T \\ & \nwarrow r_T & \downarrow \tilde{s} & \swarrow s & \\ & & \mathcal{F}(\Gamma)(v) & & \end{array}$$

It is now clear that $\tilde{s} \hat{g} i = r_T i = id$. Finally, since $\tilde{s} \hat{g} i \tilde{s} = r_T i \tilde{s} = \tilde{s}$ and \tilde{s} is injective, we have $\hat{g} i \tilde{s} = id$. \square

4 The fundamental Top-groupoid of a Top-graph

4.1 Path spaces and the fundamental Top-groupoid

For a given space X , let $\mathcal{P}X$ be the space of paths $[0, 1] \rightarrow X$ with the compact-open topology generated by subbasis sets $\langle C, W \rangle = \{\alpha \mid \alpha(C) \subseteq W\}$ where $C \subseteq [0, 1]$ is compact and $W \subseteq X$ is open. For a closed subinterval $K \subseteq [0, 1]$, let $L_K: [0, 1] \rightarrow K$ be the unique, increasing, linear homeomorphism. If $\alpha: [0, 1] \rightarrow X$ is a path, let $\alpha_K = \alpha|_K \circ L_K$ be the restricted path of α to K . If $K = \{t\} \subseteq [0, 1]$, take α_K to be the constant path $c_{\alpha(t)}$ at $\alpha(t)$. The concatenation $\alpha = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ of paths α_j such that $\alpha_{j+1}(0) = \alpha_j(1)$ is given by letting $\alpha[\frac{j-1}{n}, \frac{j}{n}] = \alpha_j$.

Consider a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ of a path α and any closed interval $K \subseteq [0, 1]$. Then $\mathcal{U}_K = \bigcap_{K \cap C_j \neq \emptyset} \langle L_K^{-1}(K \cap C_j), U_j \rangle$ is an open neighborhood of α_K . If $K = \{t\}$ is a singleton, let $\mathcal{U}_K = \langle [0, 1], \bigcap_{t \in C_j} U_j \rangle$. On the other hand, if $\beta_K = \alpha$, then $\mathcal{U}^K = \bigcap_{j=1}^n \langle L_K(C_j), U_j \rangle$ is an open neighborhood of β . If $K = \{t\}$ so that $\alpha = c_{\alpha(t)}$, let $\mathcal{U}^K = \bigcap_{j=1}^n \langle \{t\}, U_j \rangle$.

Lemma 22. *Let $\mathcal{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ be an open neighborhood in $\mathcal{P}X$ such that $\bigcup_{j=1}^n C_j = [0, 1]$. Then*

1. *For any closed interval $K \subseteq [0, 1]$, $(\mathcal{U}^K)_K = \mathcal{U} \subseteq (\mathcal{U}_K)^K$,*
2. *If $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$, then $\mathcal{U} = \bigcap_{j=1}^n (\mathcal{U}_{[t_{j-1}, t_j]})^{[t_{j-1}, t_j]}$.*

In the case that X is a **Top-graph** $|\Gamma|$, recall that vertex neighborhoods $B(x, r)$ and edge neighborhoods $U \times (a, b)$ (as in Remark 9) form a basis \mathcal{B}_Γ for the topology of $|\Gamma|$ which is closed under finite intersection. Thus sets of the form $\bigcap_{j=1}^n \langle [\frac{j-1}{n}, \frac{j}{n}], U_j \rangle$, where $\{U_j\} \subset \mathcal{B}_\Gamma$, give basis generating the topology of $\mathcal{P}|\Gamma|$, which is convenient for our purposes. An open set of this form is said to be *standard*.

The path space $\mathcal{P}|\Gamma|$ can be used to construct a **Top-graph** \mathcal{P}_Γ of paths: The object space of \mathcal{P}_Γ is the vertex set Γ_0 and $\mathcal{P}_\Gamma(x, y)$ is the subspace of $\mathcal{P}|\Gamma|$ consisting of paths from x to y . Though concatenation of paths gives a continuous operation $\cdot: \mathcal{P}_\Gamma(x, y) \times \mathcal{P}_\Gamma(y, z) \rightarrow \mathcal{P}_\Gamma(x, z)$, \mathcal{P}_Γ is not a **Top-category** because concatenation is not associative. One can obtain a **Top-category** by replacing paths with Moore paths, however, to remain consistent with [4], we refrain from doing so.

We now recall the topologically enriched version of the usual fundamental groupoid used in [4] as applied to the case of **Top-graphs**.

Definition 23. The *fundamental qTop-groupoid* of a **Top-graph** Γ is the **qTop-groupoid** $\pi^{qtop}(\Gamma, \Gamma_0) = \pi_0(\mathcal{P}_\Gamma)$ whose object space is Γ_0 and $\pi^{qtop}(\Gamma, \Gamma_0)(x, y)$ is the path component space $\pi_0(\mathcal{P}_\Gamma(x, y))$. The canonical quotient morphism is denoted $\pi: \mathcal{P}_\Gamma \rightarrow \pi^{qtop}(\Gamma, \Gamma_0)$. The *fundamental Top-groupoid* of Γ is the τ -reflection $\pi^\tau(\Gamma, \Gamma_0) = \tau(\pi^{qtop}(\Gamma, \Gamma_0))$.

The underlying groupoid of $\pi^{qtop}(\Gamma, \Gamma_0)$ and $\pi^\tau(\Gamma, \Gamma_0)$ is the familiar fundamental groupoid $\pi(|\Gamma|, \Gamma_0)$ with set of basepoints $|\Gamma|$ [5].

4.2 $\pi^\tau(\Gamma, \Gamma_0)$ is a free **Top**-groupoid

Definition 24. A path $\alpha : [0, 1] \rightarrow |\Gamma|$ is an *edge path* if $\alpha^{-1}(\Gamma_0) = \{0, 1\}$. An edge path α is *trivial* if it is a null-homotopic loop. Equivalently, an edge path is non-trivial if and only if the endpoints are distinct or if it traverses a generalized wedge of circles $\Sigma(\Gamma(x, x)_+) \subset |\Gamma|$ at a vertex x . Let \mathcal{E}_Γ be the wide sub-**Top**-graph of \mathcal{P}_Γ consisting of non-trivial edge paths.

The following Lemma is a straightforward application of the existence of Lebesgue numbers.

Lemma 25. If \mathcal{V} is an open neighborhood of an edge path α in $\mathcal{E}_\Gamma(x, y)$, then there is a standard neighborhood $\mathcal{A} = \bigcap_{j=1}^n \left\langle \left[\frac{j-1}{n}, \frac{j}{n} \right], U_j \right\rangle$, $n > 2$ of α in $\mathcal{P}|\Gamma|$ such that $\mathcal{A} \cap \mathcal{E}_\Gamma(x, y) \subseteq \mathcal{V}$ and such that U_1, U_n are vertex neighborhoods and U_2, \dots, U_{n-1} are edge neighborhoods.

Note for each edge $e \in \Gamma(x, y)$, there is a canonical non-trivial edge path $\alpha_e : [0, 1] \rightarrow |\Gamma|$ where $\alpha_e(t)$ is the image of $(e, t) \in \Gamma(x, y) \times [0, 1]$ in $|\Gamma|$.

Lemma 26. There is a canonical embedding $\Gamma^\pm \rightarrow \mathcal{E}_\Gamma$ of **Top**-graphs which induces an isomorphism $\pi_0(\mathcal{C}^\pm(\Gamma)) = \pi_0(\mathcal{C}(\Gamma^\pm)) \rightarrow \pi_0(\mathcal{C}(\mathcal{E}_\Gamma))$ of **qTop**-categories.

Proof. The embedding is the identity on objects and $e \mapsto \alpha_e$ for $e \in \Gamma(x, y)$ and $e^{-1} \mapsto \bar{\alpha}_e$ for $e^{-1} \in \Gamma(y, x)^{-1}$. Here $\bar{\beta}(t) = \beta(1 - t)$ denotes the reverse path of β .

Define a **Top**-graph morphism $eval : \mathcal{E}_\Gamma \rightarrow \Gamma^\pm$ as follows: It is the identity on vertices. If $\alpha \in \mathcal{E}_\Gamma(x, y)$, then $\alpha(1/2)$ is the image of a point $(g, s) \in (\Gamma(x, y) \sqcup \Gamma(y, x)) \times (0, 1)$ in $|\Gamma|$. Take $eval(\alpha) = g$. It is straightforward to check that $eval$ induces the inverse **qTop**-functor $\pi_0(\mathcal{C}(\mathcal{E}_\Gamma)) \rightarrow \pi_0(\mathcal{C}(\Gamma^\pm))$. \square

Since concatenation $(\alpha, \beta) \mapsto \alpha \cdot \beta$ of paths is continuous, the inclusion $\mathcal{E}_\Gamma \rightarrow \mathcal{P}_\Gamma$ gives rise to a **Top**-graph morphism $\mathcal{C}(\mathcal{E}_\Gamma) \rightarrow \mathcal{P}_\Gamma$, $\alpha_1 \alpha_2 \dots \alpha_n \rightarrow \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ on the free **Top**-category. Application of the path component space functor gives a **qTop**-functor $\phi : \pi_0(\mathcal{C}(\mathcal{E}_\Gamma)) \rightarrow \pi_0(\mathcal{P}_\Gamma) = \pi^{qtop}(\Gamma, \Gamma_0)$ to the fundamental **qTop**-groupoid.

Lemma 27. The **qTop**-functor $\phi : \pi_0(\mathcal{C}(\mathcal{E}_\Gamma)) \rightarrow \pi^{qtop}(\Gamma, \Gamma_0)$ is quotient.

Proof. Consider the following factorization of π :

$$\begin{array}{ccc} \mathcal{C}(\mathcal{E}_\Gamma) & \xleftarrow{\mathcal{D}} & \mathcal{P}_\Gamma \\ q \downarrow & & \downarrow \pi \\ \pi_0(\mathcal{C}(\mathcal{E}_\Gamma)) & \xrightarrow[\phi]{} & \pi^{qtop}(|\Gamma|, \Gamma_0) \end{array}$$

where \mathcal{D} is the decomposition morphism defined as follows. If $\alpha \in \mathcal{P}_\Gamma(x, y)$, there are finitely many intervals $[a_1, b_1], \dots, [a_n, b_n] \subset [0, 1]$ (ordered with respect to the ordering of $[0, 1]$) such that $\alpha_i = \alpha|_{[a_i, b_i]}$ is a non-trivial edge path. Now

$\mathcal{D}(\alpha)$ is defined as the word $\alpha_1\alpha_2\dots\alpha_n$ in $\mathcal{C}(\mathcal{E}_\Gamma)$. If no restriction of α is a non-trivial edge path, then α is a null-homotopic loop based at some vertex $x \in \Gamma_0$ and so we take $\mathcal{D}(\alpha)$ to be the identity id_x . Since α and $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ are homotopic paths, the diagram commutes. The decomposition morphism is a direct generalization of the decomposition function in [2, pp. 793]; it is important to note that \mathcal{D} is only a morphism of underlying algebraic graphs since it is not continuous on edge spaces.

For convenience, rename the sets $[0, a_1], [b_1, a_2], \dots, [b_{n-1}, a_n], [b_n, 1]$ (some of which may be singletons) as $K_1, K_2, \dots, K_n, K_{n+1}$. Note that each restriction α_{K_i} is a trivial loop based at a vertex x_i which has image in some vertex neighborhood $B(x_i, r_i)$. In particular, $x = x_1$ and $x_n = y$.

Given vertices $x, y \in \Gamma_0$, suppose $U \subseteq \pi_1^{qtop}(\Gamma, \Gamma_0)(x, y)$ such that $\phi^{-1}(U)$ is open in $\pi_0(\mathcal{C}(\mathcal{E}_\Gamma))(x, y)$. Thus $q^{-1}(\phi^{-1}(U))$ is open in $\mathcal{C}(\mathcal{E}_\Gamma)(x, y)$. Since π is quotient, it suffices to show $\pi^{-1}(U) = \mathcal{D}^{-1}(q^{-1}(\phi^{-1}(U)))$ is open in $\mathcal{P}_\Gamma(x, y)$. Suppose $\alpha \in \pi^{-1}(U)$ and note $\mathcal{D}(\alpha)$ lies in the open neighborhood $q^{-1}(\phi^{-1}(U))$. If $\mathcal{D}(\alpha) = id_x$ for some x , then the image of α lies in the contractible vertex neighborhood $B(x, 1)$. The neighborhood $\{\beta \in \mathcal{P}_\Gamma(x, x) | Im(\beta) \subseteq B(x, 1)\}$ of α contains only null-homotopic loops and is therefore contained in $\pi^{-1}(U)$.

On the other hand, suppose $\mathcal{D}(\alpha) = \alpha_1\dots\alpha_n \in q^{-1}(\phi^{-1}(U))$ with decomposition as describe above. In particular, $\alpha_i \in \mathcal{E}_\Gamma(x_i, x_{i+1})$. We construct a neighborhood of α contained in $\pi^{-1}(U)$. Recall that $\mathcal{B}_i = \langle [0, 1], B(x_i, r_i) \rangle$ is a neighborhood of α_{K_i} , and thus $\mathcal{B}_i^{K_i}$ is a neighborhood of α for each i . Since $\mathcal{C}(\mathcal{E}_\Gamma)$ is a **Top**-category, there are open neighborhoods V_i of α_i in $\mathcal{E}_\Gamma(x_i, x_{i+1})$ such that the product $V_1 V_2 \dots V_n$ is contained in $q^{-1}(\phi^{-1}(U))$. By Lemma 25, there is a standard neighborhood $\mathcal{A}_i = \bigcap_{j=1}^{n_i} \langle [\frac{i-1}{n_i}, \frac{i}{n_i}], U_j^i \rangle$, $n_i > 2$ of α_i in $\mathcal{P}[\Gamma]$ such that $\mathcal{U}_i = \mathcal{A}_i \cap \mathcal{E}_\Gamma(x_i, x_{i+1}) \subseteq V_i$ and such that $U_1^i, U_{n_i}^i$ are vertex neighborhoods and U_2^i, \dots, U_{n-1}^i are edge neighborhoods. In particular, choose the \mathcal{A}_i so that $U_{n_i}^i \cup U_1^{i+1} \subseteq B(x_{i+1}, r_{i+1})$.

Now

$$\mathcal{W} = \bigcap_{i=1}^n \mathcal{A}_i^{[a_i, b_i]} \cap \bigcap_{i=1}^{n+1} \mathcal{B}_i^{K_i} \cap \mathcal{P}_\Gamma(x, y)$$

is an open neighborhood of α in $\mathcal{P}_\Gamma(x, y)$.

Suppose $\beta \in \mathcal{W}$. We clearly have $\beta(K_{i+1}) \subset B(x_{i+1}, r_i)$, however if $x_i = x_{i+2}$, it is possible that $\beta_{[a_i, b_{i+1}]}$ does not hit the vertex x_{i+1} . To deal with this possibility, we replace "small" portions of β . For each $i = 2, \dots, n$, let $s_i = L_{[a_{i-1}, b_{i-1}]}(1 - \frac{1}{n_{i-1}})$ and $t_i = L_{[a_i, b_i]}(\frac{1}{n_i})$ so that $K_i = [b_{i-1}, a_i] \subset [s_i, t_i]$. Now define a path γ to equal the path β with the following exceptions: replace the portion of β from s_i to b_{i-1} with the canonical arc from $\beta(s_i)$ to x_i , take γ to be the constant at x_i on $[b_{i-1}, a_i]$, and replace the portion of β from a_i to t_i with the canonical arc from x_i to $\beta(t_i)$. Since γ is given by changing β only in contractible neighborhoods, γ and β are homotopic paths, i.e. $\pi(\beta) = \pi(\gamma)$. Moreover, $\gamma_i = \gamma_{[a_i, b_i]}$ is an edge path for each i contained in \mathcal{U}_i . Thus

$$\mathcal{D}(\gamma) = \gamma_1 \gamma_2 \dots \gamma_n \in \mathcal{U}_1 \mathcal{U}_2 \dots \mathcal{U}_n \subseteq V_1 V_2 \dots V_n \subseteq q^{-1}(\phi^{-1}(U)).$$

Finally, we see that

$$\pi(\beta) = \pi(\gamma) = \phi(q(\mathcal{D}(\gamma))) \in U$$

giving the inclusion $\mathcal{W} \subseteq \pi^{-1}(U)$. \square

Theorem 28. *The fundamental **Top**-groupoid $\pi^\tau(\Gamma, \Gamma_0)$ is naturally isomorphic to the free **Top**-groupoid $\mathcal{F}(\pi_0(\Gamma))$.*

Proof. The embedding $\Gamma \rightarrow \mathcal{P}_\Gamma$ given by $e \mapsto \alpha_e$ induces a **Top**-graph morphism $\pi_0(\Gamma) \rightarrow \pi_0(\mathcal{P}_\Gamma) = \pi^{qtop}(\Gamma, \Gamma_0)$. Additionally, the identity functor $\pi^{qtop}(\Gamma, \Gamma_0) \rightarrow \pi^\tau(\Gamma, \Gamma_0)$ is a morphism of **qTop**-groupoids. The composition $\sigma : \pi_0(\Gamma) \rightarrow \pi^\tau(\Gamma, \Gamma_0)$ of these two morphisms is a morphism of **Top**-graphs which induces a morphism $\hat{\sigma} : \mathcal{F}(\pi_0(\Gamma)) \rightarrow \pi^\tau(\Gamma, \Gamma_0)$ of **Top**-groupoids. A straightforward generalization of [3, 3.14] to **Top**-graphs with more than one vertex gives that $\hat{\sigma}$ is an isomorphism of the underlying groupoids. Therefore, it suffices to check the inverse $\hat{\sigma}^{-1} : \pi^\tau(\Gamma, \Gamma_0) \rightarrow \mathcal{F}(\pi_0(\Gamma))$ is a **Top**-functor.

Consider the following commutative diagram. The upper horizontal functors are the **qTop**-isomorphism from Lemma 26 and the canonical **qTop**-functor $\psi : \pi_0(\mathcal{C}^\pm(\Gamma)) \rightarrow \mathcal{C}^\pm(\pi_0(\Gamma))$ from Lemma 16. The vertical functor R is the quotient **qTop**-functor given by word reduction (See Remark 17) and ϕ is the quotient **qTop** functor of Lemma 27.

$$\begin{array}{ccccc} \pi_0(\mathcal{C}(\mathcal{E}_\Gamma)) & \xrightarrow{\cong} & \pi_0(\mathcal{C}^\pm(\Gamma)) & \xrightarrow{\psi} & \mathcal{C}^\pm(\pi_0(\Gamma)) \\ \phi \downarrow & & & & \downarrow R \\ \pi_1^{qtop}(\Gamma, \Gamma_0) & \xrightarrow{\hat{\sigma}^{-1}} & \mathcal{F}_R(\pi_0(\Gamma)) & & \end{array}$$

Since the top composition is a **qTop**-functor and ϕ is quotient, $\hat{\sigma}^{-1} : \pi^{qtop}(\Gamma, \Gamma_0) \rightarrow \mathcal{F}_R(\pi_0(\Gamma))$ is continuous on hom-spaces (by the universal property of quotient spaces) and is therefore a **qTop**-functor. Applying the τ -reflection gives that

$$\hat{\sigma}^{-1} : \pi^\tau(\Gamma, \Gamma_0) = \tau(\pi^{qtop}(\Gamma, \Gamma_0)) \rightarrow \tau(\mathcal{F}_R(\pi_0(\Gamma))) = \mathcal{F}(\pi_0(\Gamma))$$

is a **Top**-functor. \square

The topological group $\pi^\tau(\Gamma, \Gamma_0)(v)$ at a vertex $v \in \Gamma$ is, by definition, the topological fundamental group $\pi_1^\tau(|\Gamma|, v)$ of [3, 4]. In light of Section 3.3, we have the following Corollary.

Corollary 29. *The topological fundamental group $\pi_1^\tau(|\Gamma|, v)$ of a **Top**-graph Γ is a free Graev topological group. In particular, if Γ has more than one vertex and $T \subset \pi_0(\Gamma)$ is a maximal tree, then $\pi_1^\tau(|\Gamma|, v) \cong F_G(\pi_0(\Gamma)/T, *)$. If Γ has a single vertex, then $\pi_1^\tau(|\Gamma|, v) \cong F_G(\pi_0(\Gamma)_+, *) \cong F_M(\pi_0(\Gamma))$.*

Corollary 30. *Every free **Top**-groupoid is the fundamental **Top**-groupoid of a **Top**-graph.*

Proof. According to Remark 12, a given **Top**-graph Γ is isomorphic to $\pi_0(h(\Gamma))$ for some **Top**-graph $h(\Gamma)$. By Theorem 28, $\pi^\tau(h(\Gamma), h(\Gamma)_0) \cong \mathcal{F}(\pi_0(h(\Gamma))) \cong \mathcal{F}(\Gamma)$. \square

5 Semicoverings and a proof of Theorem 1

We recall the theory of semicovering spaces and apply it to **Top**-graphs. Our definitions and notation are those used in [4]. Given a space X and point $x \in X$, let $(\mathcal{P}X)_x$ and $(\Phi X)_x$ be spaces of paths and homotopies (rel. endpoints) of paths starting at x respectively with the compact-open topology. Let $\mathcal{P}X(x, x')$ be the subspace of $(\mathcal{P}X)_x$ consisting of paths ending at x' . In particular, $\mathcal{P}X(x, x) = \Omega(X, x)$ is the space of based loops.

Definition 31. A *semicovering* is a local homeomorphism $p : Y \rightarrow X$ such that for each $y \in Y$, the induced maps $\mathcal{P}p : (\mathcal{P}Y)_y \rightarrow (\mathcal{P}X)_{p(y)}$ and $\Phi p : (\Phi Y)_y \rightarrow (\Phi X)_{p(y)}$ are homeomorphisms. The space Y is a *semicovering (space)* of X . If α is a path or homotopy of paths starting at $p(y)$, then $\tilde{\alpha}_y$ denotes the unique lift of α starting at $y \in Y$.

Every covering (in the classical sense) is a semicovering, however, if X does not have a simply connected cover (e.g. the Hawaiian earring), a semicovering of X need not be a covering. Of particular importance to the current paper is the fact that a semicovering $p : Y \rightarrow X$ induces an open covering morphism $\pi^\tau p : \pi^\tau Y \rightarrow \pi^\tau X$. In particular, for each $y_1, y_2 \in Y$, $p(y_i) = x_i$, the induced map $p_* : \pi_1^\tau Y(y_1, y_2) \rightarrow \pi_1^\tau X(x_1, x_2)$ is an open embedding of spaces (of topological groups when $y_1 = y_2$). Just as in classical covering theory, if β is a loop based at $p(y)$, then $\tilde{\beta}_y$ is a loop based at y if and only if $[\beta]$ lies in the image of the embedding p_* . Thus, since $\pi : \mathcal{P}X(x, x') \rightarrow \pi^\tau X(x_1, x_2)$ is continuous, $\{\beta | \tilde{\beta}_{y_1}(1) = y_2\}$ is an open subspace of $\mathcal{P}X(x_1, x_2)$.

Since **Top**-graphs can fail to be locally path connected, we cannot apply classical covering theory to study them. We use the fact that semicovering theory applies to certain non-locally path connected spaces called locally wep-connected spaces [4, Definition 6.4].

Definition 32. Let X be a space.

1. A path $\alpha : [0, 1] \rightarrow X$ is *well-targeted* if for every open neighborhood \mathcal{U} of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a path $\beta \in \mathcal{U}$ with $\beta(1) = b$.
2. A path $\alpha : [0, 1] \rightarrow X$ is *locally well-targeted* if for every open neighborhood \mathcal{U} of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a well-targeted path $\beta \in \mathcal{U}$ with $\beta(1) = b$.

See also the closely related definition of *(locally) well-ended path* in [4]. A space X is *locally wep-connected* if for every pair of points $x, y \in X$, there is a locally well-targeted path from x to y .

Every locally path connected space is locally wep-connected. There are also many spaces which are locally wep-connected but not locally path connected. For example, if Γ is a **Top**-graph with a single vertex, the generalized wedge of circles $|\Gamma| = \Sigma(\Gamma_+)$ is locally wep-connected [4, Proposition 6.7] but is only locally path connected if Γ is locally path connected. The following Lemma generalizes this special case to arbitrary **Top**-graphs; the proof is nearly identical.

Lemma 33. *Every **Top**-graph is locally wep-connected.*

Proof. Since $|\Gamma|$ is locally path connected at each vertex, it suffices to find a locally well-targeted path from a vertex to each point $z \in |\Gamma| \setminus \Gamma_0$. Suppose z is the image of (e, t) in $\Gamma(x, y) \times (0, 1)$. Any path $\alpha : [0, 1] \rightarrow |\Gamma|$ such that $\alpha(0) = x$, $\alpha(1) = z$, and having image on the edge $\{e\} \times [0, 1] \subset |\Gamma|$ is locally well-targeted. The argument that α is locally well-targeted is identical to that in [4, Proposition 6.7]. \square

Since **Top**-graphs are locally wep-connected, we may apply semicovering theory to obtain the last ingredient for our proof of Theorem 1.

Lemma 34. [4, Corollary 7.20] *If Γ is a **Top**-graph, $x \in \Gamma_0$ is a vertex, and H is an open subgroup of the topological fundamental group $\pi_1^t(|\Gamma|, x)$, then there is a semicovering $p : Y \rightarrow |\Gamma|$, $p(y) = x$ such that the induced homomorphism $p_* : \pi_1^t(Y, y) \rightarrow \pi_1^t(|\Gamma|, x)$ is a topological embedding onto H .*

5.1 A semicovering of a **Top**-graph is a **Top**-graph

The following result generalizes the fact that a covering (in the classical sense) of a graph is a graph and provides the last ingredient for a proof of Theorem 1.

Theorem 35. *A semicovering of a **Top**-graph is a **Top**-graph.*

Proof. It suffices to assume the semicovering and **Top**-graph in question are connected. Let $p : Y \rightarrow |\Gamma|$ be a connected semicovering of **Top**-graph Γ . We find a **Top**-graph $\widetilde{\Gamma}$ such that $|\widetilde{\Gamma}| \cong Y$. Since Γ_0 is a discrete subspace of $|\Gamma|$ and p is a local homeomorphism, $p^{-1}(\Gamma_0)$ is a discrete subspace of Y . Define the vertex space $\widetilde{\Gamma}_0 = p^{-1}(\Gamma_0)$. For $y_1, y_2 \in \widetilde{\Gamma}_0$ such that $x_i = p(y_i)$ define

$$\widetilde{\Gamma}(y_1, y_2) = \{e \in \Gamma(x_1, x_2) \mid (\widetilde{\alpha_e})_{y_1}(1) = y_2\}$$

with the subspace topology of $\Gamma(x_1, x_2)$.

Define a map $h : |\widetilde{\Gamma}| \rightarrow Y$ as follows: The restriction of h to $\widetilde{\Gamma}_0$ is the identity. The map $h_{y_1, y_2} : \widetilde{\Gamma}(y_1, y_2) \times [0, 1] \rightarrow Y$ given by $h_{y_1, y_2}(e, t) = (\widetilde{\alpha_e})_{y_1}(t)$ is continuous since $\Gamma(x_1, x_2) \rightarrow \mathcal{P}|\Gamma|(x_1, x_2)$, $e \mapsto \alpha_e$ is continuous, $\mathcal{P}p : (\mathcal{P}Y)_{y_1} \rightarrow (\mathcal{P}X)_{x_1}$ is a homeomorphism, and evaluation $\mathcal{P}Y \times [0, 1] \rightarrow Y$, $(\beta, t) \mapsto \beta(t)$ is continuous. The maps h_{y_1, y_2} induce the function h on the image of $\widetilde{\Gamma}(y_1, y_2) \times [0, 1]$ in $|\widetilde{\Gamma}|$. It follows from Lemma 36 that h is a bijection, Lemma 37 that h is continuous, and Lemma 38 that h is an open map. Therefore h is a homeomorphism. \square

To prove the following Lemmas, we make some observations about the edge spaces $\widetilde{\Gamma}(y_1, y_2)$. Let $A = \{\alpha_e \in \mathcal{P}|\Gamma|(x_1, x_2) \mid e \in \Gamma(x_1, x_2)\}$ and recall $B = \{\beta \in \mathcal{P}|\Gamma|(x_1, x_2) \mid \beta_{y_1}(1) = y_2\}$ is open in $\mathcal{P}|\Gamma|(x_1, x_2)$. It is now clear that $A \cap B$ is open in A and is the image of $\widetilde{\Gamma}(y_1, y_2)$ under the homeomorphism $\Gamma(x_1, x_2) \cong A$, $e \mapsto \alpha_e$. Therefore, $\widetilde{\Gamma}(y_1, y_2)$ is an open subspace of $\Gamma(x_1, x_2)$. It follows that whenever $p(y_1) = x_1$,

- $\widetilde{\Gamma}_{y_1} = \Gamma_{x_1}$ and
- $\Gamma(x_1, x_2)$ decomposes as the topological sum $\coprod_{p(y_2)=x_2} \widetilde{\Gamma}(y_1, y_2)$.

Lemma 36. $h : |\widetilde{\Gamma}| \rightarrow Y$ is a bijection.

Proof. First, we show h is surjective. It suffices to consider a point $y \in Y \setminus \widetilde{\Gamma}_0$. Note $p(y)$ is the image of a pair $(e, t) \in \Gamma(x_1, x_2) \times (0, 1)$ in $|\Gamma|$. Fix $x_0 \in \Gamma_0$ and $y_0 \in p^{-1}(x_0)$ and let β be a path from y_0 to y in Y . Since $|\Gamma|$ is a connected **Top**-graph, there is a sequence of vertices $x_0, x_1, \dots, x_{n-1}, x_n$, edges $e_i \in \Gamma(x_{i-1}, x_i)$, and $\delta_i \in \{\pm 1\}$ such that $p\beta$ is homotopic (rel. endpoints) to the concatenation

$$\alpha = \alpha_{e_1}^{\delta_1} \cdot \dots \cdot \alpha_{e_n}^{\delta_n} \cdot (\alpha_e)_{[0,t]}.$$

In particular, $\tilde{\alpha}_{y_0}(1) = y$. Let y_1 be the endpoint of the lift of $\alpha_{e_1}^{\delta_1} \cdot \dots \cdot \alpha_{e_n}^{\delta_n}$ starting at y_0 and $y_2 = \tilde{\alpha}_e(1)$. By our choice of α , $p(y_i) = x_i$. Then the lift of $(\alpha_e)_{[0,t]}$ starting at y_1 ends at $h(e, t) = (\widetilde{\alpha_e})_{y_1}(t) = y$.

For injectivity, suppose $z, z' \in |\widetilde{\Gamma}|$. If one of z or z' is a vertex and $h(z) = h(z')$, then $z, z' \in \widetilde{\Gamma}_0$ and it follows that $z = z'$. Therefore it suffices to check that h is injective on $|\Gamma| \setminus \widetilde{\Gamma}_0$. Suppose z is the image of $(e, t) \in \widetilde{\Gamma}(y_1, y_2) \times (0, 1)$ and z' is the image of $(f, u) \in \widetilde{\Gamma}(y_3, y_4) \times (0, 1)$ under h . If $h(z) = (\widetilde{\alpha_e})_{y_1}(t) = (\widetilde{\alpha_f})_{y_3}(u) = h(z')$, then $\alpha_e(t) = \alpha_f(u)$ in $|\Gamma|$, however, this only occurs if $e = f$ and $t = u$ and thus $z = z'$. \square

Lemma 37. $h : |\widetilde{\Gamma}| \rightarrow Y$ is continuous.

Proof. Since the maps h_{y_1, y_2} are continuous, h is continuous if $|\Gamma|$ is given the quotient topology. However, we use the coarser topology on $|\Gamma|$ (which may differ from the quotient topology at vertices). Thus it remains to check that h is continuous at each vertex of $|\widetilde{\Gamma}|$. Suppose $y_0 \in \widetilde{\Gamma}_0$ and V is an open neighborhood of $h(y_0) = y_0$ in Y such that p maps V homeomorphically onto a vertex neighborhood $B(x_0, r)$ of $p(y_0) = x_0$. Since h is a bijection (Lemma 36),

$$h(\widetilde{B}(y_0, r)) = \{(\widetilde{\alpha_e})_{y_0}(t) \in Y \mid 0 \leq t < r, e \in \Gamma_{y_0}\} \cup \{(\widetilde{\alpha_f})_{y_0}(t) \in Y \mid 0 \leq t < r, f \in \Gamma^{y_0}\}.$$

But p maps V homeomorphically onto the path connected set $B(x_0, r)$. Therefore, the lifts of all paths in $B(x_0, r)$ starting at y_0 have image in V . It follows that $h(\widetilde{B}(y_0, r)) = V$. \square

Lemma 38. $h : |\widetilde{\Gamma}| \rightarrow Y$ is an open map.

Proof. Since $|\Gamma|$ is locally wep-connected (Lemma 33), Y is locally wep connected [4, Corollary 6.12]. Thus for each $y \in Y$, evaluation $ev_1 : (\mathcal{P}Y)_y \rightarrow Y, ev_1(\beta) = \beta(1)$ is quotient [2, Proposition 6.2]. Additionally, if $p(y) = x$, $\mathcal{P}p : (\mathcal{P}Y)_y \rightarrow (\mathcal{P}|\Gamma|)_x$ has a continuous inverse $L : (\mathcal{P}|\Gamma|)_x \rightarrow (\mathcal{P}Y)_y$. To show h is open, we use the fact that the composition $ev_1 L : (\mathcal{P}|\Gamma|)_x \rightarrow Y, \beta \mapsto \tilde{\beta}_y(1)$ is quotient whenever $p(y) = x$.

Fix a vertex $y_0 \in \widetilde{\Gamma}_0$ and $p(y_0) = x_0$. Suppose $\widetilde{B}(y_0, r)$, $0 < r < 1$ is a vertex neighborhood of $y_0 \in \widetilde{\Gamma}_0$. As in the previous lemma, if

$$V = \{(\widetilde{\alpha_e})_{y_0}(t) \in Y | 0 \leq t < r, e \in \Gamma_{y_0}\} \cup \{(\widetilde{\alpha_f})_{y_0}(t) \in Y | 0 \leq t < r, f \in \Gamma^{y_0}\},$$

then $h(\widetilde{B}(y_0, r)) = V$. Again, we use the fact that $p(V) = B(x_0, r)$ and if $\gamma : [0, 1] \rightarrow B(x_0, r)$ is the canonical arc from x_0 to a given point $z \in B(x_0, r)$, then $\widetilde{\gamma}_{y_0}$ has image in V . We claim that V is open in Y .

Since $ev_1 L$ is quotient, it suffices to show $L^{-1}(ev_1^{-1}(V))$ is open in $(\mathcal{P}|\Gamma|)_{x_0}$. If $\beta \in L^{-1}(ev_1^{-1}(V))$, then $\widetilde{\beta}_{y_0}(1) \in V$ and thus $\beta(1) \in B(x_0, r)$. Let γ be the canonical arc from x_0 to $\beta(1)$ in $B(x_0, r)$ and recall $Im(\widetilde{\gamma}_{y_0}) \subset V$. Thus $\widetilde{\beta \cdot \gamma}_{y_0}$ is a loop based at y_0 and $[\beta \cdot \gamma]$ lies in the open subgroup $p_*(\pi^\tau(Y, y_0))$ of $\pi^\tau(|\Gamma|, x_0)$. Since $\pi : \Omega(X, x_0) \rightarrow \pi^\tau(X, x_0)$ is continuous, there is a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^n \left\langle \left[\frac{j-1}{n}, \frac{j}{n} \right], U_j \right\rangle$ of $\beta \cdot \gamma$ in $\mathcal{P}|\Gamma|$ such that $\mathcal{U} \cap \Omega(X, x_0) \subseteq \pi^{-1}(p_*(\pi^\tau(Y, y_0)))$. Since $B(x_0, r)$ is contractible, we may assume 1. n is even, 2. $U_1 = B(x_0, r)$, and 3. $U_k = B(x_0, r)$ for $k \geq n/2$. Now $\mathcal{V} = \mathcal{U}_{[0, 1/2]} \cap (\mathcal{P}|\Gamma|)_{x_0}$ is an open neighborhood of β in $(\mathcal{P}|\Gamma|)_{x_0}$ which we claim is a subset of $L^{-1}(ev_1^{-1}(V))$. Suppose $\beta' \in \mathcal{V}$. Then $\beta'(1) \in U_{n/2} = B(x_0, r)$ and, if γ' is the canonical arc from x_0 to $\beta'(1)$, then $\beta \cdot \gamma' \in \mathcal{U} \cap \Omega(X, x_0) \subseteq \pi^{-1}(p_*(\pi^\tau(Y, y_0)))$. Thus $(\widetilde{\beta \cdot \gamma'})_{y_0}(1) = y_0$. Since $\widetilde{\gamma'}_{y_0}$ has image in V ,

$$ev_1 L(\beta') = \widetilde{\beta'}_{y_0}(1) = \widetilde{\gamma'}_{y_0}(1) \in V.$$

It follows that $L^{-1}(ev_1^{-1}(V))$ is open in $(\mathcal{P}|\Gamma|)_{x_0}$.

Let $y_1, y_2 \in \widetilde{\Gamma}_0$, $p(y_i) = x_i$, and suppose $U \times (a, b) \subset \widetilde{\Gamma}(y_1, y_2) \times (0, 1)$ is an edge neighborhood in $|\widetilde{\Gamma}|$. Note that

$$W = h(U \times (a, b)) = \{(\widetilde{\alpha_e})_{y_1}(t) \in Y | (e, t) \in \widetilde{\Gamma}(y_1, y_2) \times (0, 1)\}.$$

We show $L^{-1}(ev_1^{-1}(W))$ is open in $(\mathcal{P}X)_{x_1}$.

Recall $\widetilde{\Gamma}(y_1, y_2)$ is defined to be an open subspace of $\Gamma(x_1, x_2)$. Therefore ph maps $U \times (a, b)$ homeomorphically onto the corresponding edge neighborhood $U \times (a, b) \subset |\Gamma|$. If $\beta \in L^{-1}(ev_1^{-1}(W))$, then $\widetilde{\beta}_{y_1}(1) = (\widetilde{\alpha_e})_{y_1}(t) \in W$ for some $(e, t) \in U \times (a, b)$. Let $\gamma = (\alpha_e)_{[0, t]}$. Since $\widetilde{\beta \cdot \gamma}_{y_1}$ is a loop based at y_1 , we have, as in the vertex neighborhood case, that $\beta \cdot \gamma$ lies in the open neighborhood $\pi^{-1}(p_*(\pi_1(|\Gamma|, x_1))) \subset \Omega(|\Gamma|, x_1)$. Take a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^n \left\langle \left[\frac{j-1}{n}, \frac{j}{n} \right], U_j \right\rangle$ of $\beta \cdot \gamma$ in $\mathcal{P}|\Gamma|$ such that $\mathcal{U} \cap \Omega(|\Gamma|, x_1) \subseteq \pi^{-1}(p_*(\pi^\tau(|\Gamma|, x_1)))$. In particular, we may assume 1. n is even, 2. $U_1 = U_n = B(x_1, r)$ is a vertex neighborhood, 3. when $n/2 \leq k < n$, U_k is an edge neighborhood of the form $A \times (r_k, s_k)$ for an open set $A \subseteq U$, and 4. $(r_{n/2}, s_{n/2}) = (r_{(n/2)+1}, s_{(n/2)+1}) \subseteq (a, b)$. Now $\mathcal{V} = \mathcal{U}_{[0, 1/2]} \cap (\mathcal{P}|\Gamma|)_{x_1}$ is an open neighborhood of β in $(\mathcal{P}|\Gamma|)_{x_1}$. Note that $\beta(1) \in A \times (r_{n/2}, s_{n/2})$.

We check that $\mathcal{V} \subseteq L^{-1}(ev_1^{-1}(W))$. If $\beta' \in \mathcal{V}$, then $\beta'(1) \in A \times (r_{n/2}, s_{n/2}) \subseteq U \times (a, b) \subset |\Gamma|$. If $\beta'(1)$ is the image of $(f, u) \in A \times (r_{n/2}, s_{n/2}) \subseteq \Gamma(x_1, x_2) \times (0, 1)$ in $|\Gamma|$, then $\alpha_f(u) = \beta'(1)$. Choose any ϵ such that $\frac{n-2}{n} < \epsilon < 1$. Define a path γ' so that

- $\gamma'(v) = (\alpha_f)_{[0,t]}(v)$ for $v \in [0, \epsilon]$.
- $(\gamma')_{[\epsilon,1]}$ is the canonical arc from $z = (\alpha_f)_{[0,t]}(\epsilon)$ to $\alpha_f(u)$ in $A \times (r_{n/2}, s_{n/2})$.

Notice that γ' is constructed so that 1. $\beta'(1) = \gamma'(1)$, 2. $\beta' \cdot \overline{\gamma'}$ is a loop in $\mathcal{U} \subseteq \pi^{-1}(p_*(\pi_1(|\Gamma|, x_1)))$ based at x_1 , and 3. γ' is homotopic (rel. endpoints) to $(\alpha_f)_{[0,u]}$ (consequently $\widetilde{\gamma'}_{y_1}(1) = \widetilde{(\alpha_f)}_{y_1}(u)$). Since $[\beta' \cdot \overline{\gamma'}] \in p_*(\pi_1(|\Gamma|, x_1))$, we have

$$ev_1 L(\beta) = \widetilde{(\beta')}_{y_1}(1) = \widetilde{\gamma'}_{y_1}(1) = \widetilde{(\alpha_f)}_{y_1}(u) \in W.$$

Thus $\mathcal{V} \subseteq L^{-1}(ev_1^{-1}(W))$. □

5.2 A proof of Theorem 1

We conclude with a proof of Theorem 1.

Proof. Suppose X is a space with basepoint $* \in X$ and H is an open subgroup of the free topological group $F_G(X, *)$. Let $h(X)$ be a space such that $\pi_0(h(X)) = X$ (see Remark 12) and Γ be the **Top** graph with $\Gamma_0 = \{a, b\}$ (i.e. two vertices), $\Gamma(a, b) = h(X)$, and $\Gamma(b, a) = \emptyset$. Note that the edge space of $\pi_0(\Gamma)$ is precisely X . By Theorem 28, $\pi^\tau(\Gamma, \Gamma_0)$ is isomorphic to the free **Top**-groupoid $\mathcal{F}(\pi_0(\Gamma))$. A tree $T \subseteq \pi_0(\Gamma)$ is given by taking $T_0 = \{a, b\}$ with edge space $T = \{*\}$. Note $\pi_0(\Gamma)/T \cong X$ as based spaces. Theorem 21 gives the middle isomorphism in

$$\pi_1^\tau(|\Gamma|, a) = \pi^\tau(\Gamma, \Gamma_0)(a) \cong F_G(\pi_0(\Gamma)/T, *) \cong F_G(X, *).$$

By Lemma 34, there is a semicovering $p : Y \rightarrow |\Gamma|, p(y) = a$ such that the induced homomorphism $\pi_1^\tau(Y, y) \rightarrow \pi_1^\tau(|\Gamma|, a) \cong F_G(X, *)$ is a topological embedding onto H . According to Theorem 35, the semicovering space Y is a **Top**-graph. Finally, Corollary 29 applies to Y to give that $\pi_1^\tau(Y, y) \cong H$ is a free Graev topological group. □

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